

A direct algorithm of one-dimensional optimal system for the group invariant solutions

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Abstract

A direct and systematic algorithm is proposed to find one-dimensional optimal system for the group invariant solutions, which is attributed to the classification of its corresponding one-dimensional Lie algebra. Since the method is based on different values of all the invariants, the process itself can both guarantee the comprehensiveness and demonstrate the inequivalence of the optimal system, with no further proof. To illustrate our method more clearly, we give a couple of well-known examples: the Korteweg-de Vries (KdV) equation and the heat equation.

Keywords: group invariant solutions; Lie algebra; optimal system; invariants; the adjoint transformation matrix

1. Introduction

Symmetry group theory built by Sophus Lie plays an important role in constructing explicit solutions for integrable and non-integrable nonlinear equations. For any given subgroup, an original nonlinear system can be reduced to a system with fewer independent variables which corresponds to group invariant solutions. Since there are almost always an infinite amount of such subgroups, it is usually not feasible to list all possible group invariant solutions to the system. It is anticipated to find all those inequivalent group invariant solutions, that is to say, to give them a classification. The problem of classifying the subgroups and reduction to optimal systems takes on more importance for multidimensional PDEs. Given a group that leaves a PDE invariant, one desires to minimize the search for group-invariant solutions to that of finding inequivalent branches of solutions, which leads to the concept of the optimal systems. Consequently, the problem of determining the optimal system of subgroups is reduced to the corresponding problem for subalgebras. In applications, one usually constructs the optimal system of subalgebras, from which the optimal systems of subgroup and group invariant solutions are reconstructed.

The adjoint representation of a Lie group on its Lie algebra was known to Lie. Its use in classifying group-invariant solutions appears in Ovsiannikov [1]. Ovsiannikov demonstrated the construction of the one-dimensional optimal system for the Lie algebra, using a global matrix for the adjoint transformation and sketched the construction of higher-dimensional optimal systems with a simple example. The method has received extensive development by Patera, Winternitz and Zassenhaus [2, 3] and many examples of optimal systems of subgroups for the important Lie groups of mathematical physics were obtained. In the investigation of the connections between Lie group and special functions, Weisner [4] firstly gave the classification of the symmetry algebra of the heat equation. For the higher-dimensional optimal systems of Lie algebra, Galas [5] also developed Ovsiannikov's idea by removing equivalent subalgebras and the problem of a nonsolvable algebra was also discussed, which is generally harder than that for a solvable algebra. Some examples of optimal systems can also be found in Ibragimov [6, 7].

Here we are concerned with the one-dimensional optimal system of subalgebras. For the one-dimensional optimal systems, the technique of Ovsiannikov has been used until Olver gives a slightly different and elegant technique. Olver [8] constructed a table of adjoint operators to simplify a general element in Lie algebra as much as possible

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and applied the technique to the Korteweg-de Vries (KdV) equation and the heat equation. Since it only depends on fragments of the theory of Lie algebras, Olver's method as developed here has the feature of being very elementary. Based on Olver's method, we have also constructed many interesting and important invariant solutions [9–13] for a number of systems of PDEs in atmosphere and geometric field. However, as Olver said, although some sophisticated techniques are available for Lie algebras with additional structure, in essence this problem is attacked by the naïve approach. One knows that, if one calls a list of $\{\tilde{v}_\alpha\}_{\alpha \in \mathcal{A}}$ is an one-dimensional optimal system, it must satisfy two conditions: (1) completeness—any one-dimensional subalgebra is equivalent to some \tilde{v}_α ; (2) inequivalence— \tilde{v}_α and \tilde{v}_β are inequivalent for distinct α and β . Despite these numerous results on finding the representatives of subalgebras, they did not illustrate that how these representatives are comprehensive and mutually inequivalent. Recently, Chou and Qu [14–16] offer many numerical invariants to address the inequivalence among the elements in the optimal system.

The purpose of this paper is to give a systematic method for finding an optimal system of one-dimensional Lie algebra, which can both guarantee the comprehensiveness and the inequivalence. The idea is inspired by the observation that the killing form of the Lie algebra is an invariant for the adjoint representation [8]. Olver also points out that the detection of such an invariant is important since it places restrictions on how far one can expect to simplify the Lie algebra. In spite of the importance of the invariants for the Lie algebra, to the best of our knowledge, there are few literatures to use more common invariants except the killing form in the process of constructing optimal system. The purpose of this paper is to introduce a direct and valid method for providing all the general invariants which are different from the numerical invariants appearing in [14–16] and then make the best use of them with the adjoint matrix to classify subalgebras. We shall demonstrate the new technique by treating a couple of illustrative examples.

This paper is arranged as follows. In section 2, a direct algorithm of one-dimensional optimal system for the general symmetry algebra is proposed. Since the realization of our new algorithm builds on different invariants and the adjoint matrix, a valid method for computing all the invariants is also given in this section. In section 3, we apply the new algorithm to a couple of well-known examples, i.e. the Korteweg-de Vries (KdV) equation and the heat equation, and construct their one-dimensional optimal systems step by step. Finally, a brief conclusion is given in section 4.

2. A direct algorithm of one-dimensional optimal system

Consider the n -dimensional symmetry algebra \mathcal{G} of a differential system, which is generated by the vector fields $\{v_1, v_2, \dots, v_n\}$. The corresponding symmetry group of \mathcal{G} is denoted as G . Following Ovsianikov [1], one calls two elements $v = \sum_{i=1}^n a_i v_i$ and $w = \sum_{j=1}^n b_j v_j$ in \mathcal{G} equivalent if they satisfy one of the following conditions:

- (1) one can find some transformation $g \in G$ so that $Ad_g(w) = v$;
- (2) there is $v = cw$ with c being constant.

Here Ad_g is the adjoint representation of g and $Ad_g(w) = g^{-1}wg$. It needs to note that the second condition is less obvious in all the references but here it will play an important role in our method. The main tools used in our algorithm are all the invariants and the adjoint matrix.

2.1. Calculation of the invariants

A real function ϕ on the Lie algebra \mathcal{G} is called an invariant if $\phi(Ad_g(v)) = \phi(v)$ for all $v \in \mathcal{G}$ and all $g \in G$. If two vectors v and w are equivalent under the adjoint action, it is necessary that $\phi(v) = \phi(w)$ for any invariant ϕ . If we let $v = \sum_{i=1}^n a_i v_i$, then the invariant ϕ can be regarded as a function of a_1, a_2, \dots, a_n . As Olver said, the detection of such an invariant is important since it places restrictions on how far we can expect to simplify v . However, it is a pity that people did not care more invariants except the killing form. Now we will propose a valid method to find all the invariants of symmetry algebra and further make the best use of them to construct one-dimensional optimal system.

For the n -dimensional symmetry algebra \mathcal{G} , we firstly compute the commutation relations between all the vector fields v_i and v_j , which can be shown in a table, the entry in row i and column j representing $[v_i, v_j] = v_i v_j - v_j v_i$. Then taking any subgroup $g = e^w (w = \sum_{j=1}^n b_j v_j)$ to act on v , we have

$$\begin{aligned}
Ad_{exp(\epsilon w)}(v) &= e^{-\epsilon w} v e^{\epsilon w} \\
&= v - \epsilon[w, v] + \frac{1}{2!} \epsilon^2[w, [w, v]] - \dots \\
&= (a_1 v_1 + \dots + a_n v_n) - \epsilon[b_1 v_1 + \dots + b_n v_n, a_1 v_1 + \dots + a_n v_n] + o(\epsilon) \\
&= (a_1 v_1 + \dots + a_n v_n) - \epsilon(\Theta_1 v_1 + \dots + \Theta_n v_n) + o(\epsilon),
\end{aligned} \tag{1}$$

where $\Theta_i \equiv \Theta_i(a_1, \dots, a_n, b_1, \dots, b_n)$ can be easily obtained from the commutator table.

Equivalently, omitting v_i we can rewrite (1) as

$$(a_1, a_2, \dots, a_n) \xrightarrow{Ad_{exp(\epsilon w)}} (a_1 - \epsilon \Theta_1, a_2 - \epsilon \Theta_2, \dots, a_n - \epsilon \Theta_n) + o(\epsilon).$$

According to the definition of the invariant, it is necessary that

$$\phi(a_1, a_2, \dots, a_n) = \phi(a_1 - \epsilon \Theta_1 + o(\epsilon), a_2 - \epsilon \Theta_2 + o(\epsilon), \dots, a_n - \epsilon \Theta_n + o(\epsilon)) \tag{2}$$

for any b_i .

Taking the derivative of Eq. (2) with respect to ϵ and setting $\epsilon = 0$, then extracting the coefficients of all b_i , $N(N \leq n)$ linear differential equations of ϕ are obtained. By solving these equations, all the invariants can be found.

2.2. Calculation of the adjoint transformation matrix

The second task is the construction of the general adjoint transformation matrix A , which is the product of the matrices of the separate adjoint actions A_1, A_2, \dots, A_n . For further details, one can refer to Ref. [17] which showed three methods of constructing the adjoint matrix A . Here, before constructing the matrix A , one are able to draw a table, where the (i, j) -th entry gives $Ad_{exp(\epsilon v_i)}(v_j)$.

Firstly, applying the adjoint action of v_1 to $v = \sum_{i=1}^n a_i v_i$ and with the help of adjoint representation table, we have

$$\begin{aligned}
Ad_{exp(\epsilon_1 v_1)}(a_1 v_1 + a_2 v_2 + \dots + a_n v_n) \\
&= a_1 Ad_{exp(\epsilon_1 v_1)} v_1 + a_2 Ad_{exp(\epsilon_1 v_1)} v_2 + \dots + a_n Ad_{exp(\epsilon_1 v_1)} v_n \\
&= R_1 v_1 + R_2 v_2 + \dots + R_n v_n,
\end{aligned} \tag{3}$$

with $R_i \equiv R_i(a_1, a_2, \dots, a_n, \epsilon_1)$, $i = 1 \dots n$. To be intuitive, the formula (3) can be rewritten into the following matrix form:

$$v \doteq (a_1, a_2, \dots, a_n) \xrightarrow{Ad_{exp(\epsilon_1 v_1)}} (R_1, R_2, \dots, R_n) = (a_1, a_2, \dots, a_n) A_1.$$

Similarly, we can construct the matrices A_2, A_3, \dots, A_n of the separate adjoint actions of v_2, v_3, \dots, v_n , respectively. Then the general adjoint transformation matrix A is the product of A_1, \dots, A_n taken in any order

$$A \equiv A(\epsilon_1, \epsilon_2, \dots, \epsilon_n) = A_1 A_2 \dots A_n. \tag{4}$$

That is to say, applying the most general adjoint action $Ad_{exp(\epsilon_n v_n)} \dots Ad_{exp(\epsilon_2 v_2)} Ad_{exp(\epsilon_1 v_1)}$ to v , we have

$$v \doteq (a_1, a_2, \dots, a_n) \xrightarrow{Ad} (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = (a_1, a_2, \dots, a_n) A. \tag{5}$$

Remark 1: In fact, the right hand side of (5) can be regarded as n algebraic equations with respect to $\epsilon_1, \dots, \epsilon_n$, which read

$$(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = (a_1, a_2, \dots, a_n) A. \tag{6}$$

If Eqs.(6) have the solution, it means that $\sum_{i=1}^n a_i v_i$ is equivalent to $\sum_{i=1}^n \tilde{a}_i v_i$ under the adjoint action; If Eqs.(6) are incompatible, it shows that $\sum_{i=1}^n a_i v_i$ and $\sum_{i=1}^n \tilde{a}_i v_i$ are inequivalent.

2.3. the classification of \mathcal{G}

(1) **The first step:** scale the invariants.

If two vectors v and w are adjoint equivalent, it is necessary that $\phi(v) = \phi(w)$ for any invariant ϕ . However, if $v = cw$, where v and w are also equivalent, their corresponding invariants satisfy $\phi(v) = c'\phi(w)$ and it is usually $\phi(v) \neq \phi(w)$. To avoid the latter case, we firstly make a scale to the invariant by adjusting the coefficients of generators. Without loss of generality, one just need consider the values of the invariants to be 1, -1 and 0. To illustrate the point more clearly, we give three remarks.

Remark 2: If the degree of the invariant is odd, we obtain $\phi(v) = c^{2k+1}\phi(w)$ with $v = cw$, then the right c can be selected to transform the positive (negative) invariant into the negative (positive) one. Now we just need consider two cases: $\phi = 0$ and $\phi \neq 0$ (for simplicity scaling it to 1 or -1).

Remark 3: If the degree of the invariant is even (excluding zero), there is $\phi(v) = c^{2k}\phi(w)$ with $v = cw$, then we can not choose the right c to transform the positive (negative) invariant into the negative (positive) one. Now one need consider three cases: $\phi = 0$, $\phi > 0$ and $\phi < 0$. Without loss of generality, we let $\phi = 0$, $\phi = 1$ and $\phi = -1$.

Remark 4: Once one of the invariants is scaled (not zero), the other invariants (if any) can not be adjusted.

(2) **The second step:** select the representative element.

According to different values of the invariants given in step 1, select the corresponding representative element in the simplest form named $\tilde{v} = \sum_{i=1}^n \tilde{a}_i v_i$. Then solve the adjoint transformation equation (6). If Eqs.(6) have the solution with respect to $\epsilon_1, \dots, \epsilon_n$, it signifies that the selected representative element is right; If Eqs.(6) have no solution, we need reselect the proper representative element. Repeat the process until all the cases are finished in step 1.

3. the new approach for the KdV equation and the heat equation

3.1. one-dimensional optimal system for the KdV equation

The KdV equation reads

$$u_t + u_{xxx} + uu_x = 0, \quad (7)$$

which arises in the theory of long waves in shallow water and other physical systems in which both nonlinear and dispersive effects are relevant. Using the classical Lie group method, one can obtain the symmetry algebra of (7), i.e.

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = t\partial_x + \partial_u, \quad v_4 = x\partial_x + 3t\partial_t - 2u\partial_u. \quad (8)$$

Step 1: calculate the invariants.

The commutation relations between these vector fields is given by the following table, the entry in row i and column j representing $[v_i, v_j] = v_i v_j - v_j v_i$:

Table 1: the commutator table of the KdV equation

	v_1	v_2	v_3	v_4
v_1	0	0	0	v_1
v_2	0	0	v_1	$3v_2$
v_3	0	$-v_1$	0	$-2v_3$
v_4	$-v_1$	$-3v_2$	$2v_3$	0

Substituting $v = \sum_{i=1}^4 a_i v_i$ and $w = \sum_{j=1}^4 b_j v_j$ into (1), there is

$$Ad_{exp(\epsilon w)}(v) = (a_1 v_4 + \dots + a_4 v_4) - \epsilon(\Theta_1 v_1 + \dots + \Theta_4 v_4) + o(\epsilon)$$

with

$$\Theta_1 = b_1 a_4 + b_2 a_3 - b_3 a_2 - b_4 a_1, \quad \Theta_2 = 3b_2 a_4 - 3b_4 a_2, \quad \Theta_3 = -2b_2 a_4 + 2b_4 a_3, \quad \Theta_4 = 0. \quad (9)$$

For any $b_i (i = 1 \dots 4)$, we have

$$\phi(a_1, a_2, a_3, a_4) = \phi(a_1 - \epsilon\Theta_1 + o(\epsilon), a_2 - \epsilon\Theta_2 + o(\epsilon), a_3 - \epsilon\Theta_3 + o(\epsilon), a_4 - \epsilon\Theta_4 + o(\epsilon)). \quad (10)$$

In Eq. (10), taking the derivative of with respect to ϵ and setting $\epsilon = 0$, then extracting the coefficients of all b_i , four differential equations about $\phi(a_1, a_2, a_3, a_4)$ are directly obtained:

$$\begin{cases} a_4 \frac{\partial \phi}{\partial a_1} = 0, \\ a_3 \frac{\partial \phi}{\partial a_1} + 3a_4 \frac{\partial \phi}{\partial a_2} = 0, \\ a_2 \frac{\partial \phi}{\partial a_1} + 2a_4 \frac{\partial \phi}{\partial a_3} = 0, \\ a_1 \frac{\partial \phi}{\partial a_1} + 3a_2 \frac{\partial \phi}{\partial a_2} - 2a_3 \frac{\partial \phi}{\partial a_3} = 0. \end{cases} \quad (11)$$

By solving Eqs. (11), we obtain $\phi(a_1, a_2, a_3, a_4) = F(a_4)$, where F is an arbitrary function of a_4 . Here the basic invariant of the KdV equation is only one, i.e. a_4 , and a_4 is just the killing form given by Olver [8].

Step 2: calculate the adjoint matrix A.

The adjoint representation table is given as

Table 2: the adjoint representation table of the KdV equation

Ad	v_1	v_2	v_3	v_4
v_1	v_1	v_2	v_3	$v_4 - \epsilon v_1$
v_2	v_1	v_2	$v_3 - \epsilon v_1$	$v_4 - 3\epsilon v_2$
v_3	v_1	$v_2 + \epsilon v_1$	v_3	$v_4 + 2\epsilon v_3$
v_4	$e^\epsilon v_1$	$e^{3\epsilon} v_2$	$e^{-2\epsilon} v_3$	v_4

Applying the adjoint action of v_1 to

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4, \quad (12)$$

there is

$$Ad_{exp(\epsilon_1 v_1)}(a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4) = (a_1 - a_4 \epsilon_1) v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \quad (13)$$

It is easy to obtain

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon_1 & 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

Similarly, we obtain A_2, A_3 and A_4 :

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\epsilon_2 & 0 & 1 & 0 \\ 0 & -3\epsilon_2 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \epsilon_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2\epsilon_3 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ 0 & e^{3\epsilon_4} & 0 & 0 \\ 0 & 0 & e^{-2\epsilon_4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (15)$$

Then the general adjoint transformation matrix A is obtained

$$A = A_1 A_2 A_3 A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ \epsilon_3 e^{\epsilon_4} & e^{3\epsilon_4} & 0 & 0 \\ -\epsilon_2 e^{\epsilon_4} & 0 & e^{-2\epsilon_4} & 0 \\ (-\epsilon_1 - 3\epsilon_2 \epsilon_3) e^{\epsilon_4} & -3\epsilon_2 e^{3\epsilon_4} & 2\epsilon_3 e^{-2\epsilon_4} & 1 \end{pmatrix}. \quad (16)$$

Step 3: the classification of symmetry algebra (8).

According to “Remark 2”, we have two cases: $a_4 = 1$ and $a_4 = 0$. The adjoint transformation equations (6) become

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (a_1, a_2, a_3, a_4)A. \quad (17)$$

Case 1: $a_4 = 1$.

Select a representative element $\tilde{v} = v_4$. Then by solving eqs.(17), we obtain the solution

$$\epsilon_1 = a_1 - \frac{1}{3}a_2a_3, \quad \epsilon_2 = \frac{1}{3}a_2, \quad \epsilon_3 = -\frac{1}{2}a_3. \quad (18)$$

That is to say, all the $v_4 + a_1v_1 + a_2v_2 + a_3v_3$ are equivalent to v_4 .

Case 2: $a_4 = 0$.

Substituting $a_4 = 0$ into eqs.(11), we obtain a new invariant $\phi(a_1, a_2, a_3) = a_2^2a_3^3$. In terms of “Remark 2”, there are also two cases: $a_2^2a_3^3 = 1$ and $a_2^2a_3^3 = 0$.

Case 2.1: $a_2^2a_3^3 = 1$.

Adopt two representative elements $\tilde{v} = v_2 + v_3$ and $\tilde{v} = -v_2 + v_3$.

For $a_2 > 0$ and $\tilde{v} = v_2 + v_3$, eqs.(17) with $a_2^2a_3^3 = 1$ have the solution

$$\epsilon_2 = 0, \quad \epsilon_3 = -\frac{a_1}{a_2}, \quad \epsilon_4 = -\frac{1}{3}\ln(a_2). \quad (19)$$

For $a_2 < 0$ and $\tilde{v} = -v_2 + v_3$, eqs.(17) with $a_2^2a_3^3 = 1$ have the solution

$$\epsilon_2 = 0, \quad \epsilon_3 = -\frac{a_1}{a_2}, \quad \epsilon_4 = -\frac{1}{3}\ln(-a_2). \quad (20)$$

Case 2.2: $a_2^2a_3^3 = 0$.

(1) $a_3 \neq 0$ and $a_2 = 0$.

Adopt two representative elements v_3 and $-v_3$. Then $\{a_1v_1 + a_3v_3\}$ with $a_3 > 0$ is equivalent to v_3 , where the solution for eqs.(17) is $\{\epsilon_2 = \frac{a_1}{a_3}, \epsilon_4 = \frac{1}{2}\ln(a_3)\}$, while $\{a_1v_1 + a_3v_3\}$ with $a_3 < 0$ is equivalent to $-v_3$, where the solution for eqs.(17) is $\{\epsilon_2 = \frac{a_1}{a_3}, \epsilon_4 = \frac{1}{2}\ln(-a_3)\}$. Essentially, v_3 and $-v_3$ is equivalent.

(2) $a_3 = 0$: $a_2 \neq 0$ and $a_2 = 0$.

When $a_2 \neq 0$, similar to the above case (1), $a_2v_2 + a_1v_1$ is equivalent to v_2 and $-v_2$.

When $a_2 = 0$, a_1v_1 is equivalent to v_1 .

Recapitulating, an one-dimensional optimal system of symmetry algebra (8) contains

$$v_4; \quad v_3 + v_2; \quad v_3 - v_2; \quad v_3; \quad v_2; \quad v_1. \quad (21)$$

The optimal system obtained in (21) is just the same to that found by Olver [8].

3.2. one-dimensional optimal system for the heat equation

The equation for the conduction of heat in a one-dimensional road is written as

$$u_t = u_{xx}. \quad (22)$$

The Lie algebra of infinitesimal symmetries for this equation is spanned by six vector fields

$$\begin{aligned} v_1 &= \partial_x, & v_2 &= \partial_t, & v_3 &= u\partial_u, & v_4 &= x\partial_x + 2t\partial_t, \\ v_5 &= 2t\partial_x - xu\partial_u, & v_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u, \end{aligned} \quad (23)$$

and the infinitesimal subalgebra

$$v_h = h(x, t)\partial_u,$$

where $h(x, t)$ is an arbitrary solution of the heat equation. Since the infinite-dimensional subalgebra $\langle v_h \rangle$ does not lead to group invariant solutions, it will not be considered in the classification problem.

Step 1: calculate the invariants.

Now consider the six-dimensional symmetry algebra \mathcal{G} generated by $\{v_1, v_2, \dots, v_6\}$ in (23). Their commutator table is given in table 3.

Table 3: the commutator table of the heat equation

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	v_1	$-v_3$	$2v_5$
v_2	0	0	0	$2v_2$	$2v_1$	$4v_4 - 2v_3$
v_3	0	0	0	0	0	0
v_4	$-v_1$	$-2v_2$	0	0	v_5	$2v_6$
v_5	v_3	$-2v_1$	0	$-v_5$	0	0
v_6	$-2v_5$	$2v_3 - 4v_4$	0	$-2v_6$	0	0

Substituting $v = \sum_{i=1}^6 a_i v_i$ and $w = \sum_{j=1}^6 b_j v_j$ into (1), we have

$$Ad_{exp(\epsilon w)}(v) = (a_1 v_1 + \cdots + a_6 v_6) - \epsilon(\Theta_1 v_1 + \cdots + \Theta_6 v_6) + o(\epsilon)$$

with

$$\begin{aligned} \Theta_1 &= -b_4 a_1 - 2b_5 a_2 + b_1 a_4 + 2b_2 a_5, & \Theta_2 &= -2b_4 a_2 + 2b_2 a_4, & \Theta_3 &= b_5 a_1 + 2b_6 a_2 - b_1 a_5 - 2b_2 a_6, \\ \Theta_4 &= -4b_6 a_2 + 4b_2 a_6, & \Theta_5 &= -2b_6 a_1 - b_5 a_4 + b_4 a_5 + 2b_1 a_6, & \Theta_6 &= -2b_6 a_4 + 2b_4 a_6. \end{aligned} \quad (24)$$

Then consider the equation as follows

$$\phi(a_1, a_2, \dots, a_6) = \phi(a_1 - \epsilon \Theta_1 + o(\epsilon), a_2 - \epsilon \Theta_2 + o(\epsilon), \dots, a_6 - \epsilon \Theta_6 + o(\epsilon)) \quad (25)$$

for any b_i . Taking the derivative of Eq. (25) with respect to ϵ and then setting $\epsilon = 0$, extracting the coefficients of all b_i , five differential equations about $\phi(a_1, a_2, \dots, a_6)$ are directly obtained:

$$\left\{ \begin{aligned} a_4 \frac{\partial \phi}{\partial a_1} - a_5 \frac{\partial \phi}{\partial a_3} + 2a_6 \frac{\partial \phi}{\partial a_5} &= 0, \\ a_4 \frac{\partial \phi}{\partial a_2} + a_5 \frac{\partial \phi}{\partial a_1} + a_6 (2 \frac{\partial \phi}{\partial a_4} - \frac{\partial \phi}{\partial a_3}) &= 0, \\ -a_1 \frac{\partial \phi}{\partial a_1} - 2a_2 \frac{\partial \phi}{\partial a_2} + a_5 \frac{\partial \phi}{\partial a_5} + 2a_6 \frac{\partial \phi}{\partial a_6} &= 0, \\ a_1 \frac{\partial \phi}{\partial a_3} - 2a_2 \frac{\partial \phi}{\partial a_1} - a_4 \frac{\partial \phi}{\partial a_5} &= 0, \\ -a_1 \frac{\partial \phi}{\partial a_5} + (2a_2 \frac{\partial \phi}{\partial a_3} - \frac{\partial \phi}{\partial a_4}) - a_4 \frac{\partial \phi}{\partial a_6} &= 0. \end{aligned} \right. \quad (26)$$

Solving Eqs. (26), one can obtain two basic common invariants

$$\Delta_1 \equiv \phi_1(a_1, a_2, \dots, a_6) = a_4^2 - 4a_2 a_6 \quad (27)$$

and

$$\Delta_2 \equiv \phi_2(a_1, a_2, \dots, a_6) = a_4^3 + 2a_3 a_4^2 - 4a_4 a_2 a_6 + 2a_4 a_1 a_5 - 8a_2 a_3 a_6 - 2a_2 a_5^2 - 2a_1^2 a_6. \quad (28)$$

Here Δ_1 is just the famous killing form in Ref. [8] while Δ_2 is a new invariant of v which is never addressed before.

Step 2: calculate the adjoint matrix A.

The adjoint representation table is given in table 4. Applying the adjoint action of v_1 to

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6, \quad (29)$$

there is

$$\begin{aligned} Ad_{exp(\epsilon_1 v_1)}(a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6) \\ = (a_1 - a_4 \epsilon_1) v_1 + a_2 v_2 + (a_3 + a_5 \epsilon_1 - a_6 \epsilon_1^2) v_3 + a_4 v_4 + (a_5 - 2\epsilon_1 a_6) v_5 + a_6 v_6, \end{aligned} \quad (30)$$

Table 4: the adjoint representation table of the heat equation

Ad	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	v_3	$v_4 - \epsilon v_1$	$v_5 + \epsilon v_3$	$v_6 - 2\epsilon v_5 - \epsilon^2 v_3$
v_2	v_1	v_2	v_3	$v_4 - 2\epsilon v_2$	$v_5 - 2\epsilon v_1$	$v_6 - 4\epsilon v_4 + 2\epsilon v_3 + 4\epsilon^2 v_2$
v_3	v_1	v_2	v_3	v_4	v_5	v_6
v_4	$e^\epsilon v_1$	$e^{2\epsilon} v_2$	v_3	v_4	$e^{-\epsilon} v_5$	$e^{-2\epsilon} v_6$
v_5	$v_1 - \epsilon v_3$	$v_2 + 2\epsilon v_1 - \epsilon^2 v_3$	v_3	$v_4 + \epsilon v_5$	v_5	v_6
v_6	$v_1 + 2\epsilon v_5$	$v_2 - 2\epsilon v_3 + 4\epsilon v_4 + 4\epsilon^2 v_6$	v_3	$v_4 + 2\epsilon v_6$	v_5	v_6

i.e.

$$v \doteq (a_1, a_2, \dots, a_6) \xrightarrow{Ad_{exp(\epsilon_1 v_1)}} (a_1, a_2, \dots, a_6) A_1.$$

It is easy to obtain

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 & 1 & 0 \\ 0 & 0 & \epsilon_1^2 & 0 & -2\epsilon_1 & 0 \end{pmatrix}. \quad (31)$$

Similarly, A_2, \dots, A_6 are found to be

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2\epsilon_2 & 0 & 1 & 0 & 0 \\ -2\epsilon_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4\epsilon_2^2 & 2\epsilon_2 & -4\epsilon_2 & 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2\epsilon_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\epsilon_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\epsilon_4} \end{pmatrix}. \quad (32)$$

$$A_5 = \begin{pmatrix} 1 & 0 & -\epsilon_5 & 0 & 0 & 0 \\ 2\epsilon_5 & 1 & -\epsilon_5^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \epsilon_5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 2\epsilon_6 & 0 \\ 0 & 1 & -2\epsilon_6 & 4\epsilon_6 & 0 & 4\epsilon_6^2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2\epsilon_6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (33)$$

with $A_3 = E$ being the identity matrix.

Hence the general adjoint transformation matrix A is taken as

$$A = A_4 A_5 A_3 A_1 A_2 A_6 \quad (34)$$

$$= \begin{pmatrix} e^{\epsilon_4} & 0 & -\epsilon_5 e^{\epsilon_4} & 0 & 2\epsilon_6 e^{\epsilon_4} & 0 \\ 2\epsilon_5 e^{\epsilon_4} & e^{2\epsilon_4} & -(\epsilon_5^2 + 2\epsilon_6) e^{2\epsilon_4} & 4\epsilon_6 e^{2\epsilon_4} & 4\epsilon_5 \epsilon_6 e^{2\epsilon_4} & 4\epsilon_6^2 e^{2\epsilon_4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\epsilon_1 - 2\epsilon_2 \epsilon_5 & -2\epsilon_2 & \epsilon_1 \epsilon_5 + 4\epsilon_2 \epsilon_6 & 1 - 8\epsilon_2 \epsilon_6 & -2\epsilon_1 \epsilon_6 - \epsilon_5 \Xi & -2\epsilon_6 \Xi \\ -2\epsilon_2 e^{-\epsilon_4} & 0 & \epsilon_1 e^{-\epsilon_4} & 0 & -\Xi e^{-\epsilon_4} & 0 \\ 4\epsilon_1 \epsilon_2 e^{-2\epsilon_4} & 4\epsilon_2^2 e^{-2\epsilon_4} & -(\epsilon_1^2 + 2\epsilon_2 \Xi) e^{-2\epsilon_4} & 4\epsilon_2 \Xi e^{-2\epsilon_4} & 2\epsilon_1 \Xi e^{-2\epsilon_4} & \Xi^2 e^{-2\epsilon_4} \end{pmatrix}. \quad (35)$$

with $\Xi = 4\epsilon_2 \epsilon_6 - 1$.

Step 3: the classification of symmetry algebra (23).

In the following, two invariants Δ_1 and Δ_2 will be made full use of to give an classification of the algebra \mathcal{G} . Since the degree of $\Delta_1 = a_4^2 - 4a_2 a_6$ is two, we can scale it to three cases: $\Delta_1 = 1$, $\Delta_1 = -1$ and $\Delta_1 = 0$ according to “*Remark 3*”.

Case 1: $\Delta_1 = a_4^2 - 4a_2 a_6 = 1$, $\Delta_2 = c$.

Here c is an arbitrary real constant. Under $\Delta_1 = 1$ and $\Delta_2 = c$, choose a representative element, for example, select $\tilde{v} = v_4 + \frac{c-1}{2}v_3$ (i.e. $\tilde{a}_4 = 1, \tilde{a}_3 = \frac{c-1}{2}$).

From $\Delta_1 = a_4^2 - 4a_2a_6 = 1$, we know that a_2, a_4, a_6 can not be all zeros simultaneously. Without loss of generality, one only considers $a_6 \neq 0$. For $a_6 = 0$ ($a_2 \neq 0$ or $a_4 \neq 0$), one can transform it into the case of $a_6 \neq 0$ by selecting the appropriate ϵ_i ($i = 1 \cdots 6$) which are shown in eqs.(6).

For $a_6 \neq 0$, the general solution of $\Delta_1 = 1$ and $\Delta_2 = c$ is

$$a_2 = \frac{a_4^2 - 1}{4a_6}, \quad a_3 = a_1^2a_6 - a_1a_4a_5 - \frac{1}{2}a_4 + \frac{c}{2} - \frac{a_4^2 - 1}{4a_6}a_5^2. \quad (36)$$

where a_1, a_4, a_5, a_6 are arbitrary real constants. According to the formula (6), six algebra equations about ϵ_i are proposed. After substituting $\tilde{a}_4 = 1, \tilde{a}_3 = \frac{c-1}{2}, \tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0$ with (36) into these equations, one can find the solution:

$$\epsilon_1 = \frac{a_5 + 2a_1a_4a_6 - a_4^2a_5}{2a_6}e^{\epsilon_4}, \quad \epsilon_2 = \frac{a_4 - 1}{4a_6}e^{2\epsilon_4}, \quad \epsilon_5 = (2a_1a_6 - a_4a_5)e^{-\epsilon_4}, \quad \epsilon_6 = -\frac{1}{2}a_6e^{-2\epsilon_4}. \quad (37)$$

Case 2: $\Delta_1 = a_4^2 - 4a_2a_6 \equiv -1, \Delta_2 = c$.

From $\Delta_1 = -1$, it illustrates $a_6 \neq 0$. Now the relation among a_i reads

$$a_2 = \frac{a_4^2 + 1}{4a_6}, \quad a_3 = -a_1^2a_6 + a_1a_4a_5 - \frac{1}{2}a_4 - \frac{c}{2} - \frac{a_4^2 + 1}{4a_6}a_5^2. \quad (38)$$

When $a_6 > 0$ and $a_6 < 0$, take the representative element $\tilde{v} = \frac{1}{2}(v_2 + v_6 - cv_3)$ and $\tilde{v} = \frac{1}{2}(-v_2 - v_6 - cv_3)$ respectively. Then Eqs.(6) with (38) are separately proved right by selecting

$$\epsilon_1 = -\frac{\sqrt{2}}{2} \frac{2a_1a_4a_6 - a_5 - a_4^2a_5}{\sqrt{a_6}}, \quad \epsilon_2 = \frac{1}{2}a_4, \quad \epsilon_4 = \frac{1}{2}\ln(2a_6), \quad \epsilon_5 = -\frac{\sqrt{2}}{2} \frac{2a_1a_6 - a_4a_5}{\sqrt{a_6}}, \quad \epsilon_6 = 0 \quad (39)$$

and

$$\epsilon_1 = -\frac{1}{2} \frac{\sqrt{-2a_6}(2a_1a_4a_6 - a_5 - a_4^2a_5)}{a_6}, \quad \epsilon_2 = -\frac{1}{2}a_4, \quad \epsilon_4 = \frac{1}{2}\ln(-2a_6), \quad \epsilon_5 = \frac{1}{2} \frac{\sqrt{-2a_6}(2a_1a_6 - a_4a_5)}{a_6}, \quad \epsilon_6 = 0. \quad (40)$$

In this case, the general one-dimensional Lie algebra (29) is equivalent to $v_2 + v_6 + \beta v_3$ with β being arbitrary.

Case 3: $\Delta_1 = 0, \Delta_2 = c$.

Notice that $\Delta_1 = 0$ and Δ_2 itself is an odd polynomial with respect to a_i , so one just need to consider $\Delta_2 = 1$ and $\Delta_2 = 0$ according to “Remark 2”.

Case 3.1: $\Delta_1 = 0, \Delta_2 = 1$.

Select a representative element $\tilde{v} = -v_2 - \frac{\sqrt{2}}{2}v_5$.

When $a_6 \neq 0$, there must be $a_6 < 0$ for the identity $2a_6 = -(2a_1a_6 - a_4a_5)^2$ solved by $\Delta_1 = 0$ and $\Delta_2 = 1$. Under the restriction of invariants, we have

$$a_2 = \frac{a_4^2}{4a_6}, \quad a_1 = \frac{a_4a_5 + \sqrt{-2a_6}}{2a_6} \quad \text{and} \quad a_2 = \frac{a_4^2}{4a_6}, \quad a_1 = \frac{a_4a_5 - \sqrt{-2a_6}}{2a_6}. \quad (41)$$

Then after choosing

$$\begin{aligned} \epsilon_1 &= \frac{\sqrt{2}}{4} \cdot \frac{e^{\epsilon_4}}{a_6} (\sqrt{2}a_5 + e^{\epsilon_4} + \sqrt{2}a_4\epsilon_5e^{\epsilon_4}), & \epsilon_2 &= \frac{e^{\epsilon_4}}{4a_6} (\mp 2\sqrt{-a_6} + a_4e^{\epsilon_4}), \\ \epsilon_5 &= \pm \frac{\sqrt{2}}{8} \cdot \frac{e^{-\epsilon_4}}{\sqrt{-a_6}} (4a_4a_6 + 2a_5^2 + 8a_3a_6 + e^{2\epsilon_4}), & \epsilon_6 &= \pm \frac{1}{2} \sqrt{-a_6}e^{-\epsilon_4}, \end{aligned} \quad (42)$$

one can transform (29) with (41) into $\tilde{v} = -v_2 - \frac{\sqrt{2}}{2}v_5$.

When $a_6 = 0$, one can act by any $Ad_{exp(\epsilon_6 v_6)}$ with $\epsilon_6 \neq 0$ to get a nonzero coefficient in front of v_6 , reducing to the previous case.

Case 3.2: $\Delta_1 = 0, \Delta_2 = 0$.

Case 3.2.1: Not all a_2, a_4 and a_6 are zeros. Without loss of generality, we just consider the case of $a_6 \neq 0$.

Substituting $\Delta_1 = 0$ and $\Delta_2 = 0$ into Eqs.(26), we obtain a new invariant

$$\Delta_3 = 4a_3 + 2a_4 + \frac{a_5^2}{a_6}. \quad (43)$$

Then there are now two cases, depending on the sign of the invariant Δ_3 :

(1) $\Delta_3 = 1$. Solving $\Delta_3 = 1$, we have

$$a_1 = \frac{a_4 a_5}{2a_6}, \quad a_2 = \frac{a_4^2}{4a_6}, \quad a_3 = \frac{a_6 - a_5^2 - 2a_4 a_6}{4a_6}. \quad (44)$$

When $a_6 > 0$, choose the representative element $\tilde{v} = \frac{1}{4}v_3 + v_6$, then Eqs.(6) with (44) have the solution

$$\epsilon_1 = \frac{a_5 \sqrt{a_6} + a_4 a_6 \epsilon_5}{2a_6}, \quad \epsilon_2 = \frac{a_4}{4}, \quad \epsilon_4 = \frac{1}{2} \ln a_6. \quad (45)$$

When $a_6 < 0$, the representative element is taken as $\tilde{v} = \frac{1}{4}v_3 - v_6$. It is easy to see that Eqs.(6) are right with

$$\epsilon_1 = \frac{a_5 \sqrt{-a_6} - a_4 a_6 \epsilon_5}{2a_6}, \quad \epsilon_2 = -\frac{a_4}{4}, \quad \epsilon_4 = \frac{1}{2} \ln(-a_6). \quad (46)$$

Here it is noted that $\frac{1}{4}v_3 + v_6$ and $\frac{1}{4}v_3 - v_6$ is inequivalent.

(2) $\Delta_3 = 0$. Now we have

$$a_1 = \frac{a_4 a_5}{2a_6}, \quad a_2 = \frac{a_4^2}{4a_6}. \quad (47)$$

It can be easily proved that via the same adjoint transformation (45) and (46), the Lie algebra (29) is converted into v_6 and $-v_6$, respectively.

Case 3.2.2: $a_2 = a_4 = a_6 = 0$. Substituting $a_2 = a_4 = a_6 = 0$ into (6), we find that it can also be divided into two cases:

(1) Not all a_1 and a_5 are zeros. Here we suppose $a_5 \neq 0$.

When $a_5 \neq 0$, give a representative element v_1 . One can see that Eqs.(6) with $a_2 = a_4 = a_6 = 0$ has a solution

$$\epsilon_1 = \frac{e^{\epsilon_4}(e^{\epsilon_4} \epsilon_5 a_1 - a_3)}{a_5}, \quad \epsilon_2 = \frac{1}{2} \frac{e^{\epsilon_4}(e^{\epsilon_4} a_1 - 1)}{a_5}, \quad \epsilon_6 = -\frac{1}{2} a_5 e^{-\epsilon_4}. \quad (48)$$

(2) $a_1 = a_5 = 0$. Now we have $a_1 = a_2 = a_4 = a_5 = a_6 = 0$ and the general Lie algebra (29) becomes v_3 .

In summary, an optimal system of one-dimensional subalgebras of the heat equation is found to be those spanned by

$$\begin{aligned} \omega_1(\alpha) &= v_4 + \alpha v_3, & \alpha &\in \mathbb{R}, \\ \omega_2(\beta) &= v_2 + v_6 + \beta v_3, & \beta &\in \mathbb{R}, \\ \omega_3 &= v_2 + \frac{\sqrt{2}}{2} v_5, \\ \omega_4 &= \frac{1}{4} v_3 + v_6, \\ \omega_5 &= \frac{1}{4} v_3 - v_6, \\ \omega_6 &= v_6, \\ \omega_7 &= v_1, \\ \omega_8 &= v_3. \end{aligned} \quad (49)$$

The resulting optimal system (49) of the heat equation is really optimal and completely equivalent to that given in Ref. [14], which is a further reduction to the result of Olver [8].

Remark 5: The key point of our new method is to solve some algebra equations which are reflected in (6) and it can easily be carried out by Maple.

4. Summary and discussion

Group invariant solutions have been used to great effect in the description of the asymptotic behaviour of much more general solutions to systems of partial differential equations. These group invariant solutions are characterized by their invariance under some symmetry group of the system of partial differential equations. Since there are almost always an infinite number of different symmetry groups one might employ to find group invariant solutions, a means of determining which groups give fundamentally different types of invariant solutions is essential for gaining a complete understanding of the solutions which might be available. This classification problem can be solved by looking at the adjoint representation of the symmetry group on its Lie algebra, which firstly used by Ovsiannikov. The one-dimensional classification of the symmetry algebras of the KdV equation and the heat equation are demonstrated by Olver with an easy-to-operate method in detail, which only depends on the fragments of the theory of Lie algebras. However, as Olver said, in essence this problem is attacked by the naïve approach of taking a general element in Lie algebra and subjecting it to various adjoint transformations so as to “simplify” it as much as possible. To make up this problem and ensure the comprehensiveness with inequivalence, we develop a direct and systemic algorithm for the one-dimensional optimal system. The new approach is very natural and every elements in the optimal system can be found step by step.

Our method introduced in this paper, which is essentially new, only depends on the commutator and adjoint representative relations among the generators of Lie algebras. The main work includes:

- (1) A valid method is proposed to compute all the general invariants of the one-dimensional Lie algebra, which include the well-known killing form;
- (2) A criterion is introduced to scale the invariants, which appears in “Remark 2”, “Remark 3” and “Remark 4”;
- (3) For two one-dimensional subalgebras ν and $\tilde{\nu}$, we introduce an algebraic equations system (6) to determine their equivalences in the sense of adjoint transformation;
- (4) Based on all the scaled invariants, we put forward a direct and effective algorithm to construct one-dimensional optimal system. With the new approach, every element in the optimal system can be found step by step.

Since all the representative elements are attached to different values of the invariants, it ensures the optimality of the optimal system. From the process of the operation in our method, one can easily see that how these representatives are mutually inequivalent. Due to the optimal system of symmetry algebra, a family of group invariant solutions can be recovered. Since many important equations arising from physics are of low dimensions and reducing them to ODEs requires only the determination of small parameter optimal systems, we hope this method will be useful elsewhere. Furthermore, how to apply all the invariants to construct r -parameter ($r \geq 2$) optimal systems is in our consideration. Since the algorithm is very systemic, we believe that it will provide a very good manner for the mechanization.

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